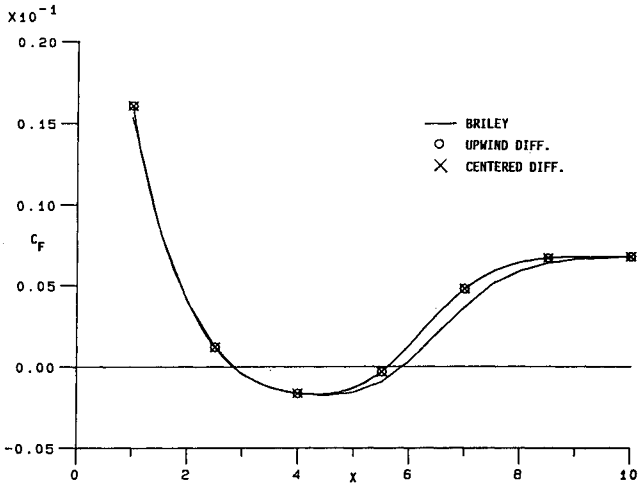

Fig. 2 Effect of discretization method on skin-friction distribution for separated flat flow. Uniform 61 × 31 grid and R = 1024.

efficiency gives similar results as for the external flow problem. The combination of line relaxation of the coupled equations and direct solution of the stream-function equation (Method III) turns out to be about two times faster than the Newton's method with direct solution of both stream-function and vorticity equations (Method IV) and approximately six times faster than line relaxation of the coupled equations only (Method I). Again, the different schemes generate virtually identical solutions.

Conclusions

Using Newton's linearization and Gaussian elimination with partial pivoting to solve the finite-difference equations of the coupled system for ψ and ω simultaneously results in a very robust algorithm, but it requires large storage. However, for two-dimensional problems, this storage requirement is affordable even on a VAX computer. This fully implicit calculation converges quadratically, provided a meaningful initial guess is used. The algorithm is particularly attractive if unstructured grids (finite volumes or finite elements) are used. (For unstructured grids the full Navier-Stokes equations are needed.) Applications of Newton's method and direct solver to three-dimensional problems in general are not possible on the present computers.

The combination of a direct solver for the stream-function equation and line-relaxation method for the coupled stream-function and vorticity equations results in the fastest scheme. This hybrid method requires less memory than the solution technique based on Newton's method and direct solution of ψ and ω simultaneously. The LU decomposition of the linear stream-function discrete equations is performed once. Therefore, the subsequent calculations are very inexpensive. Extension of this hybrid scheme for solving three-dimensional flow problems is promising.

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Transonic Flows with Vorticity Transport Around Slender Bodies

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I. Introduction

IN the last decade, methods of modeling vortical incompressible flows around slender bodies have been developed. A review of these methods is given in Ref. 1. The basic premise of these methods is that discrete vortices are introduced into an otherwise potential flow. If the discrete vortices are excluded from the domain, then the flow velocities in the domain can be considered as the superposition of potential flow velocities and the velocities induced by the vortices. The formulation requires that the governing equations be linear (to allow superposition) and, hence, excludes nonlinear compressible flows such as transonic flow. Also the vortex elements must be tracked, and this can become a complicated computational procedure.

This Note is concerned with the derivation of a technique for compressible flows that is similar to that for incompressible flows. The vorticity transport equations are derived from Crocco's equation, and for slender bodies it is found that the flow is isentropic to a first approximation and that only the crossflow vorticity is significant. The latter result is similar to one used in the incompressible theory. The present theory does not require discrete vortices but computes a vorticity field, thus avoiding the need for tracking the vortex elements. In the incompressible limit the "standard" formulation is recovered, and hence the present theory can be regarded as a unifying theory for all speed ranges.

Analysis

The Euler equations for steady compressible flow are

$$(\rho U)_x + (\rho V)_y + (\rho W)_z = 0 \tag{1}$$

$$(\rho U^2 + p)_x + (\rho UV)_y + (\rho UW)_z = 0 \tag{2}$$

$$(\rho UV)_x + (\rho V^2 + p)_y + (\rho VW)_z = 0 \tag{3}$$

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$$(\rho UW)_x + (\rho VW)_y + (\rho W^2 + p)_z = 0 \tag{4}$$

$$[\rho U(h + 1/2q^2)]_x + [\rho V(h + 1/2q^2)]_y + [\rho W(h + 1/2q^2)]_z = 0 \tag{5}$$

$$q^2 = U^2 + V^2 + W^2 \tag{6}$$

where ρ is the density, U, V, and W are velocity components in the Cartesian coordinate system (x,y,z), h is the specific enthalpy, and p is the pressure.

Manipulation of the Euler equations and the use of Gibbs relation leads to Crocco's equation

$$\bar{\mathbf{q}} \times \bar{\Omega} = T\,\overline{\nabla} S \tag{7}$$

where S is entropy, T the temperature, and $\bar{q}$ the velocity vector given by

$$\bar{q} = \bar{i}U + \bar{j}V + \bar{k}W \tag{8}$$

$$\overline{\nabla} = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z} \tag{9}$$

The vorticity vector $\bar{\Omega}$ is defined by

$$\bar{\Omega} = \overline{\nabla} \times \bar{\mathbf{q}} \tag{10}$$

Equation (7) can be differentiated to give

$$\overline{\nabla} \times (\bar{q} \times \bar{\Omega}) = \overline{\nabla} T \times \overline{\nabla} S \tag{11}$$

or, using Eq. (7),

$$\overline{\nabla} \times (\bar{q} \times \bar{\Omega}) = \overline{\nabla} T \times (\bar{q} \times \bar{\Omega})/T \tag{12}$$

In component form Eq. (12) can be written as

$$U\Omega_{1x} + (V\Omega_1)_y + (W\Omega_1)_z = \Omega_2 U_y + \Omega_3 U_z + [T_y(U\Omega_2 - V\Omega_1) - T_z(W\Omega_1 - U\Omega_3)]/T \tag{13}$$

$$(U\Omega_2) + V\Omega_{2y} + (W\Omega_2)_z = \Omega_1 V_x + \Omega_3 V_z + [T_x(U\Omega_2 - V\Omega_1) - T_z(V\Omega_3 - W\Omega_2)]/T \tag{14}$$

$$(U\Omega_3)_x + (V\Omega_3)_y + W\Omega_{3z} = \Omega_1 W_x + \Omega_2 W_y + [T_x(U\Omega_3 - W\Omega_1) - T_y(W\Omega_2 - V\Omega_3)]/\mathrm{T} \tag{15}$$

where

$$\bar{\Omega} = \bar{i}\Omega_1 + \bar{j}\Omega_2 + \bar{k}\Omega_3$$

Assume that the vorticity is produced on a slender body where the thickness to length ratio is characterized by the small parameter ϵ. Thus, the dimensions of the body in the y and z directions are of order ϵ. In order to make the dimensions of the body equal, the following tranformation is used:

$$\left.\begin{aligned} \tilde{x} &= x \\ \tilde{y} &= y/\epsilon \\ \tilde{z} &= z/\epsilon \end{aligned}\right\} \tag{16}$$

In addition, it is assumed that the velocity components U V, W can be expanded in the usual slender-body expansion to give

$$\left.\begin{aligned} U &= U_\infty(1 + \epsilon^2 u) \\ V &= \epsilon U_\infty v \\ W &= \epsilon U_\infty w \end{aligned}\right\} \tag{17}$$

The temperature T is also expanded as a series; thus

$$T = T_\infty(1 + \epsilon^m T_1) \tag{18}$$

where $m \geq 1$. Using Eqs. (16–18), it can be shown that a first approximation to Eqs. (13–15) is

$$\Omega_{1\tilde{x}} + (v\Omega_1)_{\tilde{y}} + (w\Omega_1)_{\tilde{z}} = (T_{1\tilde{y}}\Omega_2 + T_{1\tilde{z}}\Omega_3)\epsilon^{m-1} \tag{19}$$

$$\Omega_{2\tilde{x}} + v\Omega_{2\tilde{y}} + (w\Omega_2)_{\tilde{z}} = \epsilon\Omega_1 v_{\tilde{x}} + \Omega_3 v_{\tilde{z}} \tag{20}$$

$$\Omega_{3\tilde{x}} + (v\Omega_3)_{\tilde{y}} + w\Omega_{3\tilde{z}} = \epsilon\Omega_1 w_{\tilde{x}} + \Omega_2 w_{\tilde{y}} \tag{21}$$

If at some boundary the vorticity that is initiated has a vector in the x direction, then Eqs. (19–21) show that to a first approximation Ω_2 and Ω_3 are negligible in comparison with Ω_1, which is then given by

$$\Omega_{1\tilde{x}} + (v\Omega_1)_{\tilde{y}} + (w\Omega_1)_{\tilde{z}} = 0 \tag{22}$$

Thus, in the slender-body approximation one component, the crossflow vorticity, is dominant to a first approximation, and this vorticity is transported throughout the fluid without the interchanging with the other components. The neglected terms are of the order $\epsilon\Omega_1$. In order to solve Eq. (2), it is necessary to specify the boundary conditions. These boundary conditions are the location of the separation line and the magnitude of the shed vorticity. These must be found from empirical relations such as those used by Mendenhall and Perkins[1].

Assume that the velocity field is composed of an irrotational part, denoted by the subscript i, and a rotational part, denoted by the subscript r. Assume also that only the Ω_1 component of vorticity is significant, i.e., terms of order $\epsilon\Omega_1$ are negligible. The vorticity equations then become (dropping the superscript tilde in the following for convenience)

$$V_{iz} - W_{iy} = 0, \qquad V_{rz} - W_{ry} = \Omega_1 \tag{23a}$$

$$U_{iy} - V_{ix} = 0, \qquad U_{ry} - V_{rx} = \Omega_2 \simeq 0 \tag{23b}$$

$$U_{iz} - W_{ix} = 0, \qquad U_{rz} - W_{rx} = \Omega_3 \simeq 0 \tag{23c}$$

In Eqs. (23b) and (23c) the equations for the rotational components simply duplicate the irrotational component, and it suggests that a velocity potential ϕ exists such that

$$U_i = \phi_x, \qquad V_i = \phi_y, \qquad W_i = \phi_z \tag{24}$$

and that

$$U_r \simeq 0(\epsilon V_r)$$

thus to first-order

$$U_r = 0 \tag{25}$$

A vector potential A is defined as

$$\bar{q}_r = \overline{\nabla} \times \bar{A} \tag{26}$$

where

$$\bar{q}_r = \bar{i}U_r + \bar{j}V_r + \bar{k}W_r \tag{27}$$

$$\bar{A} = \bar{i}A_1 + \bar{j}A_2 + \bar{k}A_3 \tag{28}$$

Substituting Eqs. (27) and (28) into Eq. (23) gives

$$\begin{aligned} A_{1yy} + A_{1zz} &= -\Omega_1 \\ A_2 &= 0 \\ A_3 &= 0 \end{aligned} \tag{29}$$

$$V_r = A_{1z}, \qquad W_r = -A_{1y} \tag{30}$$

Since

$$U_r = A_2 = A_3 = 0$$

the subscript "1" will be omitted in the following discussion.

The equations governing the transport of vorticity are Eq. (22) and the following equations:

$$A_{yy} + A_{zz} = \Omega_1 \tag{31}$$

$$\left.\begin{aligned} U &= \phi_z \\ V &= \phi_y - A_z \\ W &= \phi_z + A_y \end{aligned}\right\} \tag{32}$$

where ϕ is the velocity potential.

In the present formulation, the rotational velocity components V_r and W_r can be regarded as quasi-two-dimensional since Eq. (31) involves only y and z derivatives. However, there is an x variation because Ω_1 varies with x because of Eq. (22).

The standard transonic potential wing theory can be deduced from Eqs. (1) and (5) with the irrotational assumption and the isentropic relation.

$$p/\rho^\gamma = p_\infty/\rho_\infty^\gamma \tag{33}$$

Equation (5) can be written, using Eqs. (1) and (32), as

$$\left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} = 1 + (\gamma - 1)\frac{M_\infty^2}{2}(1 - U^2 - V^2 - W^2) \tag{34}$$

Using Eq. (32), Eq. (34) becomes

$$\left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} = 1 + (\gamma - 1)\frac{M_\infty^2}{2}\left[1 - \phi_x^2 - (\phi_y + A_z)^2 - (\phi_z - A_y)^2\right] \tag{35}$$

Equation (35) is an equation for ρ in terms of ϕ, A. The set of equations, Eq. (1), the irrotational equations, and Eq. (35) with

$$A_z = A_y = 0$$

are the equations solved by the traditional potential method. In order to solve for a flow with vorticity, two additional equations, namely Eq. (22) and (31), must be solved. Equation (22) gives the vorticity transport and Eq. (31) the rotational velocity induced by the vorticity.

If a slender-body approximation is made to Eqs. (1) and (35), the x derivative will vanish, giving only a crossflow; this would be consistent with the use of Eq. (22). However, it is usually essential to retain the x derivatives in a transonic calculation in order to resolve shock waves, and the only consistent conditions under which the slender-body approximations to the vorticity transport and Eqs. (1) and (35) can be combined are if the terms retained in the derivation of Eq. (22) are also retained in Eq. (35). If the streamwise component of vorticity Ω_1 is the same order of magnitude as $(\rho U)_x$, then to a first approximation Eq. (22) is consistent with the use of Eqs. (1) and (35). Thus, if

$$O \mid \Omega_1 \mid \approx O \mid (\rho U)_x \mid \tag{36}$$

then the transonic flow over a slender body with first-order vorticity effects is given by Eq. (1), (22), (31), and (35) together with the irrotational relations.

Relationship to the Incompressible Discrete Vortex Model

Over the last decade, models of separated flow using discrete vortices have been developed for incompressible flow, and it is instructive to compare the present formulation with this discrete vortex model.

For incompressible flow the governing equations, Eqs. (1), (31), and (22), reduce to

$$\nabla^2\phi = 0 \tag{37}$$

$$A_{yy} + A_{zz} = -\Omega \tag{38}$$

$$\frac{\partial\Omega}{\partial x} + v\frac{\partial\Omega}{\partial y} + w\frac{\partial\Omega}{\partial z} = 0 \tag{39}$$

Since the streamwise velocity U is replaced by its freestream value, Eq. (39) denotes the transport of Ω along particle paths. That is, a discrete vortex element introduced at some point in space is unchanged in strength as it is transported throughout the flowfield.

If, at a streamwise point x, the vortex element is located at (y,z), then at $x + \Delta x$ the element is at

$$(y + \Delta y, z + \Delta z)$$

Now, in the present formulation

$$\Delta y = \frac{\Delta y}{\Delta x}\Delta x = \frac{\Delta y}{\Delta t}\frac{\Delta t}{\Delta x}\Delta x = \frac{V}{U}\Delta x = \epsilon v \Delta x \tag{40}$$

$$\Delta z = \frac{\Delta z}{\Delta x}\Delta x = \frac{\Delta z}{\Delta t}\frac{\Delta t}{\Delta x}\Delta x = \frac{W}{U}\Delta x = \epsilon w \Delta x \tag{41}$$

where U, V, W are the velocity components of the fluid particle containing the vortex element. Thus, Eqs. (40) and (41) give the location of the vortex element at the streamwise station $x + \Delta x$.

The velocity induced by the vortex elements is found from Eqs. (38) and (39). Equation (38) can be written in integral form using Green's theorem and gives

$$V_r = A_z = \int_C (K_z A_n - K_{nz} A)\, dC + \int_D\int K_z \Omega \, d\eta \, d\zeta \tag{42}$$

$$W_r = -A_y = -\int_C (K_y A_n - K_{ny} A)\, dC - \int_D\int K_y \Omega \, d\eta \, d\zeta \tag{43}$$

where

$$K = \frac{1}{4\pi}\ln\left[(y - \eta)^2 + (z - \zeta)^2\right] \tag{44}$$

and D is the two-dimensional crossflow domain excluding the body boundary C. The coordinate direction n is the inward-drawn normal to C.

The first integral on the right-hand side of Eqs. (42) and (43) is a boundary term and is evaluated by requiring that there be no flow through the body. The double integral denotes the velocity components induced by an infinite number of vortices of strength $\Omega d\eta d\zeta$. If the integral Eq. (42) is discretized to give

$$\int_D\int K_z \Omega \, d\eta \, d\zeta \simeq \lim_{N \to \infty} \sum_{i=1}^{N} K_z(\eta_i, \zeta_i, y, z)\gamma_i \tag{45}$$

where γ_i is the strength $(\Omega_i \Delta\eta \Delta\zeta)$ of the vortex at location i (η_i, ζ_i), then it may be seen that this integral is equivalent to the result obtained using the Biot-Savart law for the induced velocity in the y coordinate due to a collection of N discrete vortices. A similar result applies to Eq. (43).

In summary, the incompressible limit of the present theory gives the velocity and, consequently, the pressure on a body by the superposition of the velocity found from a solution of Laplace's equation and the crossflow velocity induced by a finite number of discrete vortices. The vorticity is constant for a particular fluid element and is transported with the fluid. This model of rotational flow is identical to that used by Mendenhall and Perkins[1] and others for incompressible flow. The present theory, therefore, gives a framework for incorporating vorticity into the classic potential equation for incompressible and compressible flow.

Concluding Remarks

A formulation for vorticity effects has been derived for slender bodies and incorporated into a compressible flow model. The formulation reduces to the standard equations for incompressible flow and, hence, provides a unifying framework for all speed ranges.

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Machine Intelligence and Autonomy for Aerospace Systems

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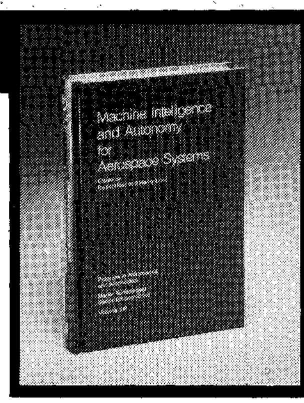

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